Math 155, ***Lecture Notes- Bonds*** **Name____________**

Section 8.7 *Indeterminate Forms and L'Hôpital's Rule*

Back in the Chapter 1 we applied algebraic methods for dealing with the following limits:

$$\lim_{x\to -1}\frac{2x^2-2}{x+1} \text{ and } \lim_{x\to\infty}\frac{3x^2-1}{2x^2+1}.$$

In the first limit, if you "plugged-in" -1 for x you would get $\frac{0}{0}$, and in the second limit if we "plugged-in" ∞ for x you would get $\frac{\infty}{\infty}$. Of course, these "plug-in" methods are not legitimate and the corresponding results are not legitimate. However, both $\frac{0}{0}$ and $\frac{\infty}{\infty}$ are called **indeterminate forms**. Both of these situations are examples of competing/conflicting interests, or rules, and it's not clear which will win out.

When we consider the limit with the form $\frac{0}{0}$, we might think of a fraction that has a numerator of zero as being zero, yet at the same time, we might think of fractions in which the denominator is going to zero as a type of infinity, or that the limit might not exist. On the other hand, we might think of a fraction in which the numerator and denominator are the same and that the limit value is one. What is really going on? Can we understand which interest will rule? Is it possible that the competing/conflicting interests will "cancel out," and the limit will reach some other value?

When we consider the indeterminate form $\frac{-\infty}{\infty}$, we run into similar issues. That is, when the numerator of a fraction is tending to negative infinity, we might think of the whole fraction tending to negative infinity, yet if the denominator is going to infinity we might think of the whole fraction as tending to zero. On the other hand, we might think of a fraction in which the numerator and denominator are the same (discounting the minus sign) and so we might believe that the limit is -1. What is really going on? Can we understand which interest will rule?

When we consider the indeterminate forms like $\frac{-\infty}{\infty}$, we run into another problem in that infinity isn't really a number, and we really shouldn't even treat it like a number. Frequently, we will find that it won't behave as we would expect it to if it was a number. In fact, this unpredictability is the mail problem with indeterminate forms. At first glance, cannot tell what is happening in the limit. We will need to investigate each situation.

By the way, we will study five other types of indeterminate forms as well.

$$0^0,\ 1^\infty,\ \infty^0,\ \pm\infty\cdot 0, \text{ and } \infty-\infty$$

Each of these indeterminate forms exhibit competing/conflicting interests, and it's just not clear which, if any, of the interests or rules will determine the value of the limit under consideration. In addition to considering graphs of functions and tables of function values, we will use **L'Hôpital's Rule** to investigate the behaviors of indeterminate forms. At times, some creative algebra will be employed to aid our study.

THEOREM 8.4 L'Hôpital's Rule

Let f and g be functions that are differentiable on an open interval (a, b) containing c, except possibly at c itself. Assume that $g'(x) \neq 0$ for all x in (a, b), except possibly at c itself. If the limit of $f(x)/g(x)$ as x approaches c produces the indeterminate form $0/0$, then

$$\lim_{x\to c} \frac{f(x)}{g(x)} = \lim_{x\to c} \frac{f'(x)}{g'(x)}$$

provided the limit on the right exists (or is infinite). This result also applies if the limit of $f(x)/g(x)$ as x approaches c produces any one of the indeterminate forms ∞/∞, $(-\infty)/\infty$, $\infty/(-\infty)$, or $(-\infty)/(-\infty)$.

This theorem tells us that when we consider an indeterminate form that is $\frac{0}{0}$, or $\frac{\pm\infty}{\pm\infty}$, we can differentiate the numerator, differentiate the denominator, and then consider the resulting limit.

Ex.1 Evaluate: $\lim_{x\to 2} \frac{\sin(x-2)}{2x-4}$

$$\lim_{x\to 2} \frac{\sin(x-2)}{2x-4} = \frac{\sin[(2)-2]}{2(2)-4} = \frac{\sin(0)}{0} = \boxed{\frac{0}{0}}$$

Stop! Indeterminate Form

$$\lim_{x\to 2} \frac{\sin(x-2)}{2x-4} = \boxed{\lim_{x\to 2} \frac{\frac{d}{dx}[\sin(x-2)]}{\frac{d}{dx}[2x-4]}}$$

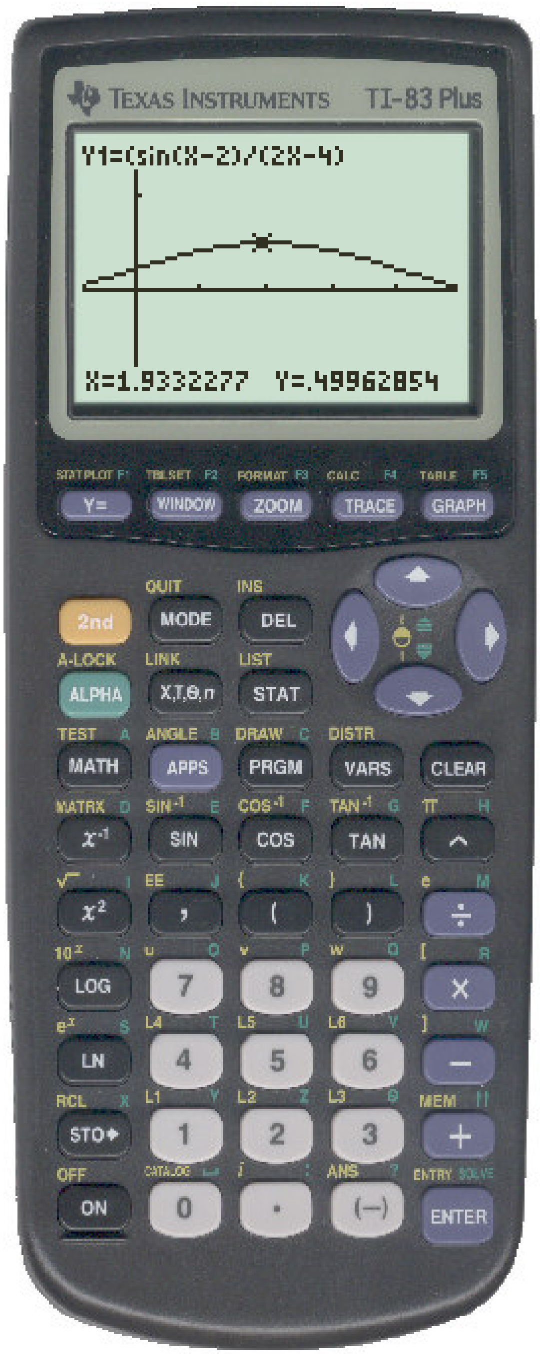
← Using L'Hôpital's Rule

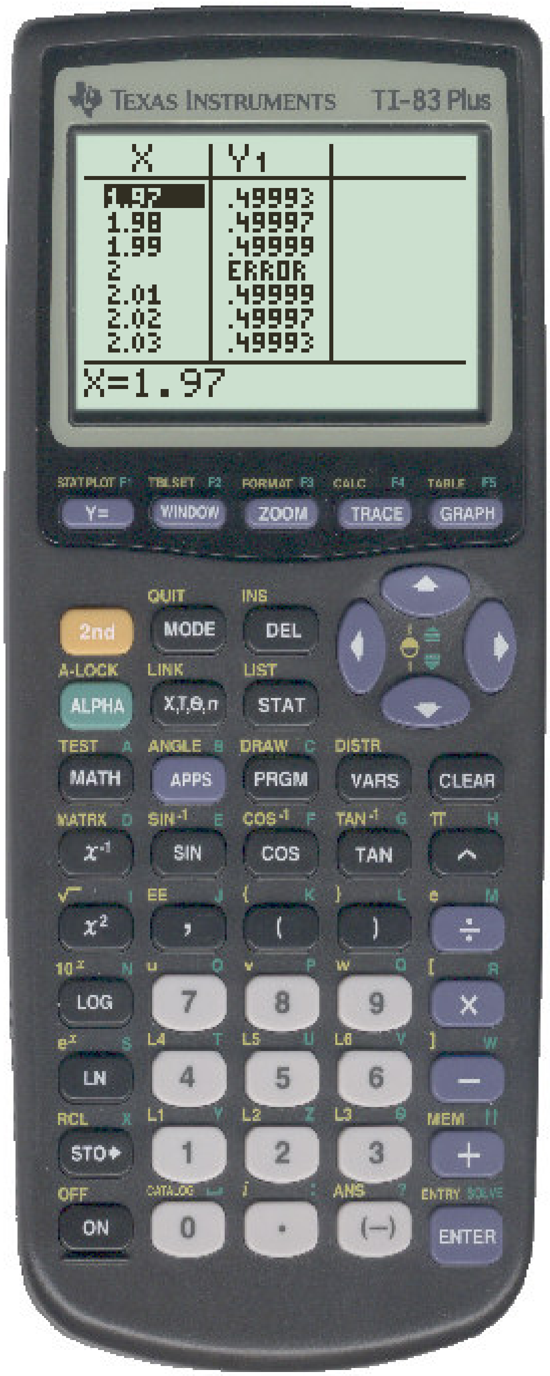
$$= \lim_{x\to 2} \frac{\cos(x-2)\cdot 1}{2}$$

$$= \frac{\cos[(2)-2]}{2}$$

$$= \frac{\cos(0)}{2}$$

$$= \frac{1}{2}$$


TEXAS INSTRUMENTS TI-83 Plus
Y1=(sin(X-2)/(2X-4)
X=1.9332277 Y=.49962854


TEXAS INSTRUMENTS TI-83 Plus
X Y1
1.97 .49993
1.98 .49997
1.99 .49999
2 ERROR
2.01 .49999
2.02 .49997
2.03 .49993
X=1.97

Ex.2 Evaluate: $\displaystyle\lim_{x\to\infty}\frac{10x^2+3x+7}{2x^2-6}$

$$\lim_{x\to\infty}\frac{10x^2+3x+7}{2x^2-6}=\frac{10(\infty)^2+3(\infty)+7}{2(\infty)^2-6}=\frac{\infty}{\infty}$$

Stop! Indeterminate form

$$\lim_{x\to\infty}\frac{10x^2+3x+7}{2x^2-6}=\lim_{x\to\infty}\frac{\frac{d}{dx}[10x^2+3x+7]}{\frac{d}{dx}[2x^2-6]}$$

← Using L'Hôpital's Rule

$$=\lim_{x\to\infty}\frac{20x+3}{4x}\longrightarrow \text{still } \frac{\infty}{\infty}$$

Indeterminate form

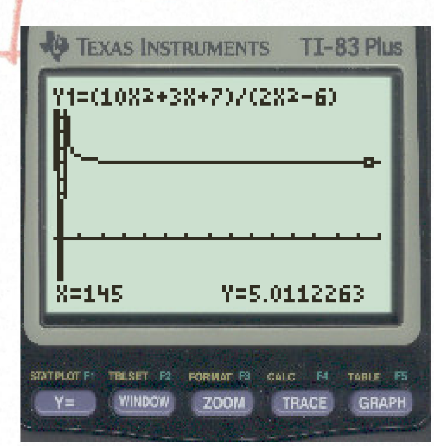
$$=\lim_{x\to\infty}\frac{\frac{d}{dx}[20x+3]}{\frac{d}{dx}[4x]}$$

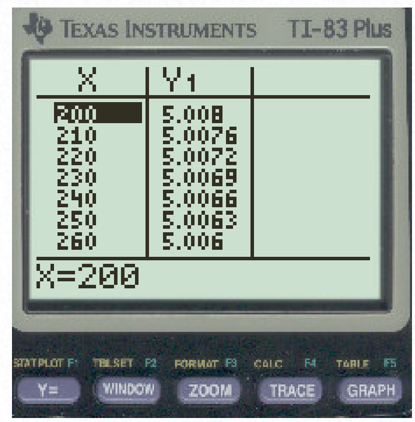
← Using L'Hôpital's Rule

$$=\lim_{x\to\infty}\frac{20}{4}$$

$$=\lim_{x\to\infty}5$$

$$=5 \checkmark$$





Ex.3 Evaluate: $\displaystyle\lim_{x\to\infty}\frac{\ln(x^4)}{x^3}$

$$\lim_{x\to\infty}\frac{\ln(x^4)}{x^3}=\frac{\ln[(\infty)^4]}{(\infty)^3}=\frac{\infty}{\infty}$$

Stop! Indeterminate form

OR

$$\lim_{x\to\infty}\frac{\ln(x^4)}{x^3}=\lim_{x\to\infty}\frac{4\ln(x)}{x^3}=\frac{4\ln(\infty)}{(\infty)^3}=\frac{\infty}{\infty}$$

Stop!

$$=4\cdot\lim_{x\to\infty}\frac{\frac{d}{dx}[\ln(x)]}{\frac{d}{dx}[x^3]}$$

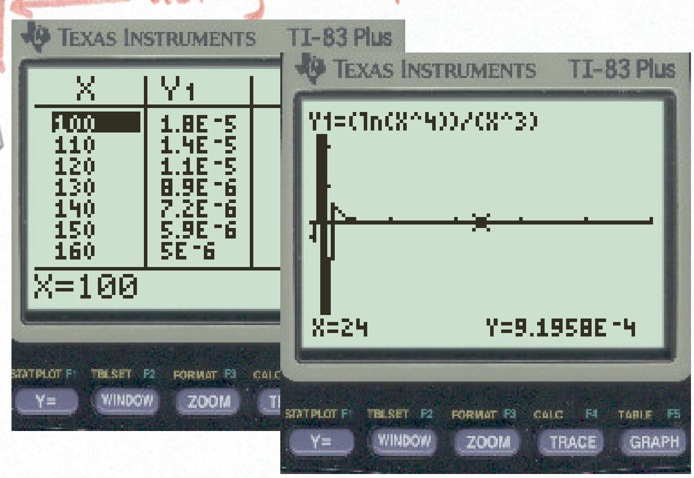
← Using L'Hôpital's Rule

$$=4\cdot\lim_{x\to\infty}\left[\frac{\frac{1}{x}}{3x^2}\right]$$

$$=\frac{4}{3}\lim_{x\to\infty}\left(\frac{1}{x^3}\right)$$

$$=\frac{4}{3}\cdot(0)$$

$$=0 \checkmark$$



Ex.4 Evaluate: $\lim\limits_{x \to 0^+} \dfrac{e^x - (1+x)}{x^3}$

$$\lim_{x \to 0^+} \frac{e^x - (1+x)}{x^3} = \frac{e^{(0)} - [1+(0)]}{(0)^3} = \frac{1-1}{0} = \boxed{\frac{0}{0}}$$

stop! Indeterminate Form

$$\lim_{x \to 0^+} \frac{e^x - (1+x)}{x^3} = \boxed{\lim_{x \to 0^+} \frac{\frac{d}{dx}[e^x - (1+x)]}{\frac{d}{dx}[x^3]}}$$

← using L'Hôpital's Rule

$$= \lim_{x \to 0^+} \frac{e^x - 1}{3x^2} \longrightarrow \text{still } \boxed{\frac{0}{0}}$$

Indeterminate Form

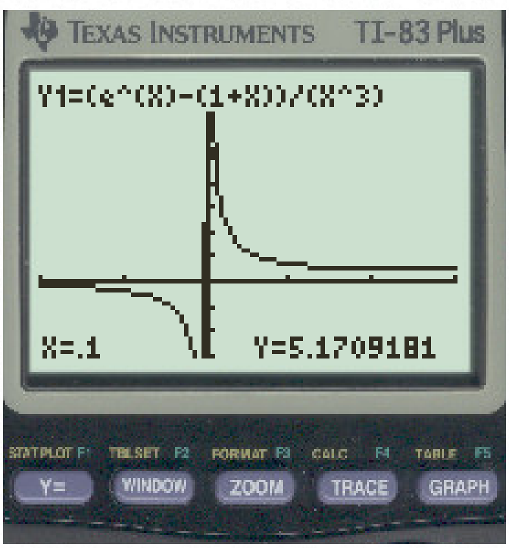
$$= \boxed{\lim_{x \to 0^+} \frac{\frac{d}{dx}[e^x - 1]}{\frac{d}{dx}[3x^2]}}$$

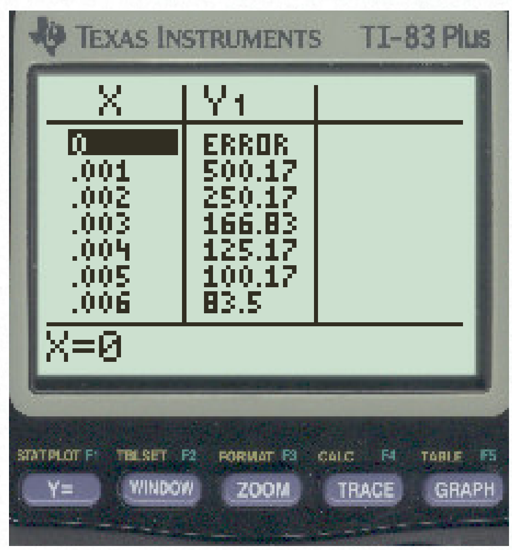
← Using L'Hôpital's Rule

$$= \lim_{x \to 0^+} \frac{e^x}{6x} \longrightarrow \frac{1}{0}$$

" $\frac{1}{0}$ " is Not an Indeterminate Form

$$= \infty \checkmark$$





Ex.5 Evaluate: $\displaystyle\lim_{x\to 2^-}\frac{\sqrt{4-x^2}}{x-2}$

$$\lim_{x\to 2^-}\frac{\sqrt{4-x^2}}{x-2} = \frac{\sqrt{4-(2)^2}}{(2)-2} = \boxed{\frac{0}{0}}$$

Stop! Indeterminate Form

$$\lim_{x\to 2^-}\frac{\sqrt{4-x^2}}{x-2} \doteq \boxed{\lim_{x\to 2^-}\frac{\frac{d}{dx}\left[(4-x^2)^{1/2}\right]}{\frac{d}{dx}\left[x-2\right]}}$$

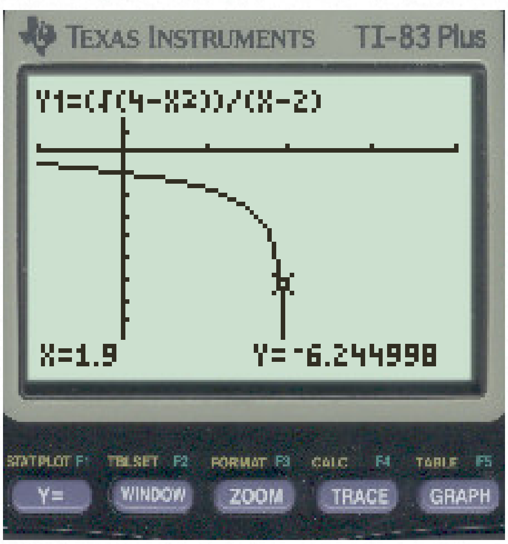
← Using L'Hôpital's Rule

$$= \lim_{x\to 2^-}\frac{\frac{1}{2}(4-x^2)^{-1/2}\cdot(-2x)}{1}$$

$$= \lim_{x\to 2^-}\frac{-x}{\sqrt{4-x^2}}$$

" $\frac{-2}{0}$ " is Not an Indeterminate Form

$$= -\infty$$



Ex.6 Evaluate: $\lim_{x\to\infty} x\cdot\tan\left(\frac{1}{x}\right)$

$$\lim_{x\to\infty} x\cdot\tan\left(\tfrac{1}{x}\right) = (\infty)\cdot\tan\left(\tfrac{1}{\infty}\right) = \boxed{\infty\cdot 0}$$

Stop! Indeterminate Form

Not Ready for L'Hôpital's Rule

$$\lim_{x\to\infty} x\cdot\tan\left(\tfrac{1}{x}\right) = \lim_{x\to\infty}\left[\frac{x\cdot\tan\left(\frac{1}{x}\right)}{1}\right]\cdot\left[\frac{\frac{1}{x}}{\frac{1}{x}}\right]$$

$$= \lim_{x\to\infty}\frac{\tan\left(\frac{1}{x}\right)}{\frac{1}{x}} \longrightarrow \frac{\tan\left(\frac{1}{\infty}\right)}{\frac{1}{\infty}} = \frac{\tan(0)}{0} = \boxed{\frac{0}{0}}$$

Indeterminate form ✓

$$= \boxed{\lim_{x\to\infty}\frac{\frac{d}{dx}\left[\tan(x^{-1})\right]}{\frac{d}{dx}\left[x^{-1}\right]}}$$

using L'Hôpital's Rule

$$= \lim_{x\to\infty}\frac{\left[\sec^2(x^{-1})\right]\cdot\left[-1\cdot x^{-2}\right]}{\left[-1\cdot x^{-2}\right]}$$

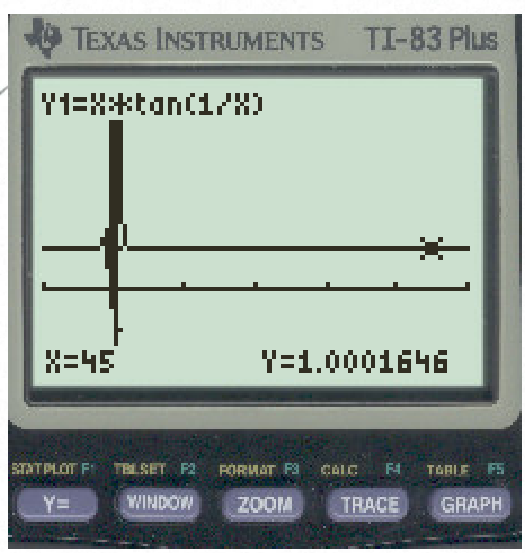
$$= \lim_{x\to\infty}\sec^2\left(\tfrac{1}{x}\right)$$

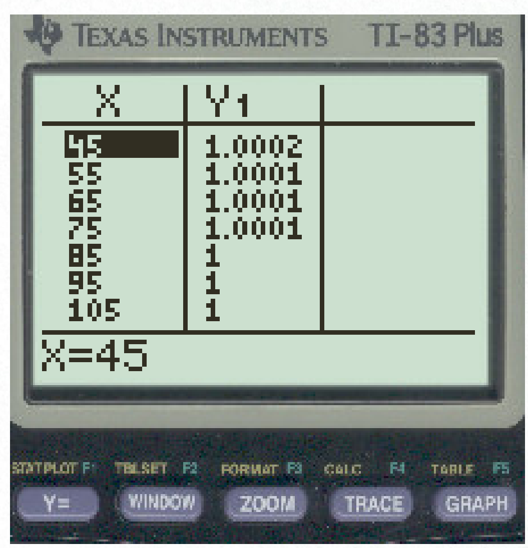
$$= \sec^2\left[\lim_{x\to\infty}\left(\tfrac{1}{x}\right)\right]$$

★★★ Limits pass through composition of continuous functions!

$$= \sec^2(0)$$

$$= \underline{1}\ \checkmark$$





Ex.7 Evaluate: $\lim_{x\to 0}\left[\frac{1}{x\sin(x)}-\frac{1}{x^2}\right]$

$$\lim_{x\to 0}\left[\frac{1}{x\sin(x)}-\frac{1}{x^2}\right]=\frac{1}{0}-\frac{1}{0}=\boxed{\infty-\infty}$$ Stop! Indeterminate Form

$$\lim_{x\to 0}\left[\frac{1}{x\sin(x)}-\frac{1}{x^2}\right]=\lim_{x\to 0}\left[\frac{x}{x}\cdot\frac{1}{x\sin(x)}-\frac{\sin(x)}{\sin(x)}\cdot\frac{1}{x^2}\right]$$

$$=\lim_{x\to 0}\left[\frac{x-\sin(x)}{x^2\sin(x)}\right]\longrightarrow\frac{0-\sin(0)}{(0)^2\sin(0)}=\boxed{\frac{0}{0}}$$ Stop! Indeterminate Form

$$=\boxed{\lim_{x\to 0}\frac{\frac{d}{dx}[x-\sin(x)]}{\frac{d}{dx}[x^2\sin(x)]}}$$ ← Using L'Hôpital's Rule

$$=\lim_{x\to 0}\frac{1-\cos(x)}{x^2\cos(x)+2x\sin(x)}\longrightarrow\frac{1-\cos(0)}{(0)^2\cos(0)+2(0)\sin(0)}=\boxed{\frac{0}{0}}$$ Stop! Indeterminate Form

$$=\boxed{\lim_{x\to 0}\frac{\frac{d}{dx}[1-\cos(x)]}{\frac{d}{dx}[x^2\cos(x)+2x\sin(x)]}}$$ ← Using L'Hôpital's Rule

$$=\lim_{x\to 0}\frac{-(-\sin(x))}{x^2(-\sin(x))+2x\cos(x)+2x\cos(x)+2\sin(x)}$$

$$=\lim_{x\to 0}\frac{\sin(x)}{-x^2\sin(x)+4x\cos(x)+2\sin(x)}\longrightarrow\frac{\sin(0)}{-(0)^2\sin(0)+4(0)\cos(0)+2\sin(0)}$$

$$=\frac{0}{0+0+0}=\boxed{\frac{0}{0}}$$ Stop! Indeterminate Form

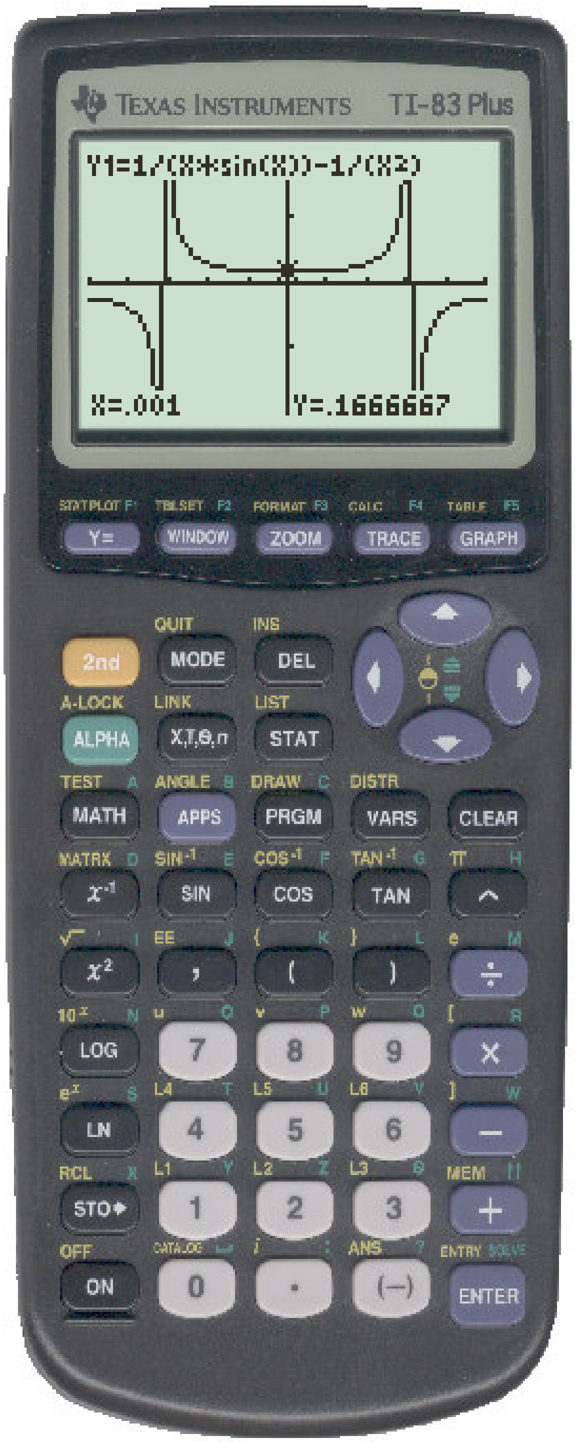
$$=\boxed{\lim_{x\to 0}\frac{\frac{d}{dx}[\sin(x)]}{\frac{d}{dx}[-x^2\sin(x)+4x\cos(x)+2\sin(x)]}}$$ ← Using L'Hôpital's Rule

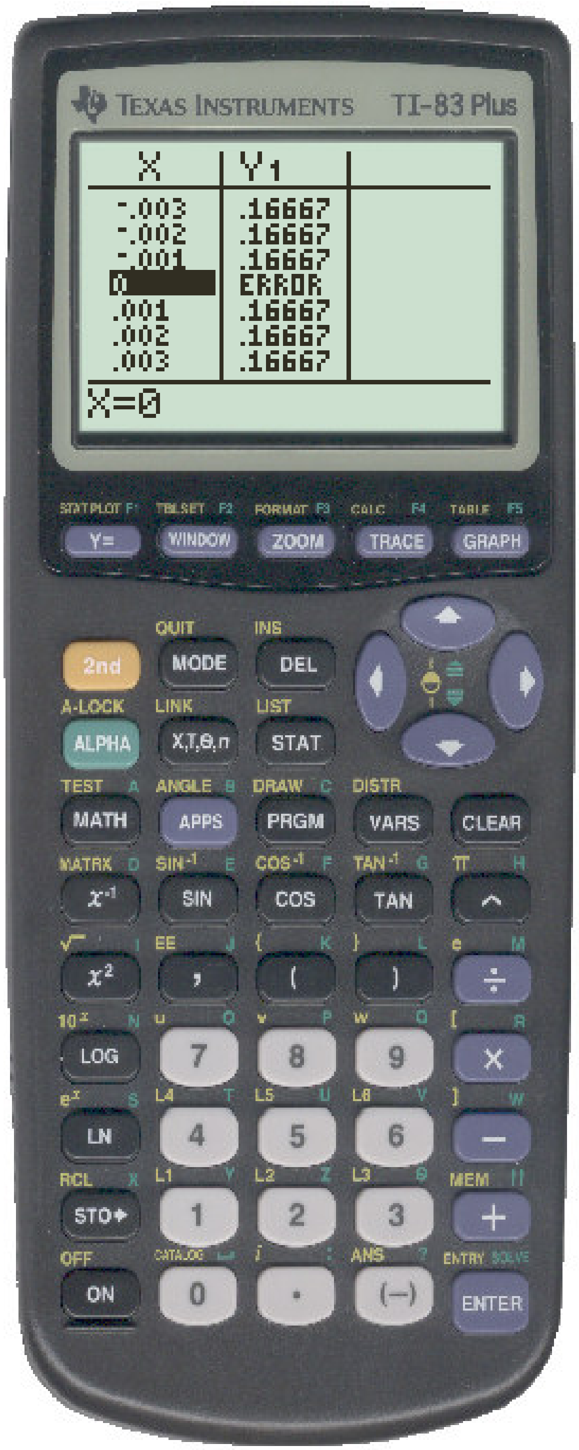
$$=\lim_{x\to 0}\frac{\cos(x)}{-x^2\cos(x)-2x\sin(x)+4x(-\sin(x))+4\cos(x)+2\cos(x)}$$

$$=\lim_{x\to 0}\frac{\cos(x)}{-x^2\cos(x)-6x\sin(x)+6\cos(x)}$$

$$=\frac{\cos(0)}{-(0)^2\cos(0)-6(0)\sin(0)+6\cos(0)}$$

$$=\frac{1}{6}$$ ✓


TEXAS INSTRUMENTS TI-83 Plus
Y1=1/(X*sin(X))-1/(X²)
X=.001
Y=.1666667


TEXAS INSTRUMENTS TI-83 Plus
X
Y1
-.003 .16667
-.002 .16667
-.001 .16667
0 ERROR
.001 .16667
.002 .16667
.003 .16667
X=0

Notice the variable in the Exponent!

Ex.8 Evaluate: $\lim_{x\to\infty} x^{\frac{1}{x}}$

$\lim_{x\to\infty} x^{\frac{1}{x}} = \infty^{\left(\frac{1}{\infty}\right)} = \boxed{\infty^0}$ stop! Indeterminate Form

Let $y = \lim_{x\to\infty} x^{\frac{1}{x}}$

$\ln(y) = \ln\left[\lim_{x\to\infty} x^{\frac{1}{x}}\right]$ ★★★

$\ln(y) = \lim_{x\to\infty}\left[\ln\left(x^{\frac{1}{x}}\right)\right]$ Use log rules to bring the exponent "down."

$\ln(y) = \lim_{x\to\infty}\left[\frac{1}{x}\cdot\ln(x)\right]$

$\ln(y) = \lim_{x\to\infty}\left[\frac{\ln(x)}{x}\right] \longrightarrow \frac{\ln(\infty)}{(\infty)} = \boxed{\frac{\infty}{\infty}}$ stop! Indeterminate Form

$\ln(y) = \boxed{\lim_{x\to\infty}\frac{\frac{d}{dx}[\ln(x)]}{\frac{d}{dx}[x]}}$ ← Using L'Hôpital's Rule

$\ln(y) = \lim_{x\to\infty}\frac{\frac{1}{x}}{1}$

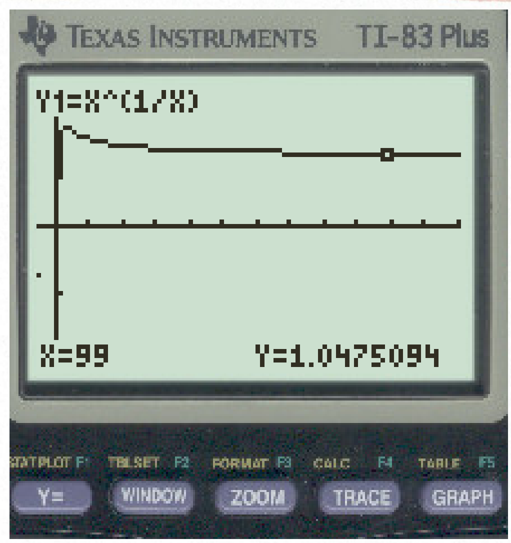
$\ln(y) = \lim_{x\to\infty}\frac{1}{x}$

$\ln(y) = 0$

$e^{\ln(y)} = e^0$

$y = 1$

So, $\lim_{x\to\infty} x^{\frac{1}{x}} = 1$ ✓



X	Y1
500	1.0125
550	1.0115
600	1.0107
650	1.01
700	1.0094
750	1.0089
800	1.0084

X=800

Apply Trig. Identity

Ex.9 Evaluate: $\lim_{x\to 0^+}\left[\cos\left(\frac{\pi}{2}-x\right)\right]^x = \lim_{x\to 0^+}[\sin(x)]^x$

Notice the variable in the exponent!

> **Trig Identity**
> $\cos(\pi/2 - x) = \sin(x)$
> $\sin(\pi/2 - x) = \cos(x)$

$\lim_{x\to 0^+}[\sin(x)]^x = \boxed{0^0}$ Stop! Indeterminate Form

Let $y = \lim_{x\to 0^+}[\sin(x)]^x$

$\ln(y) = \ln\left(\lim_{x\to 0^+}[\sin(x)]^x\right)$ ← ★★★

$\ln(y) = \lim_{x\to 0^+}\left\{\ln\left([\sin(x)]^x\right)\right\}$

Use Log rules to bring the exponent "down."

$\ln(y) = \lim_{x\to 0^+}\left(x\cdot\ln[\sin(x)]\right)$

Not Ready for L'Hôpital's Rule

$\ln(y) = \lim_{x\to 0^+}\left(\frac{x\cdot\ln[\sin(x)]}{1}\right)\cdot\left[\frac{\frac{1}{x}}{\frac{1}{x}}\right]$

$\ln(y) = \lim_{x\to 0^+}\frac{\ln[\sin(x)]}{\frac{1}{x}} \rightarrow \frac{\ln[\sin(0)]}{\frac{1}{0}} = \boxed{\frac{-\infty}{\infty}}$ Stop! Indeterminate Form

$\boxed{\ln(y) = \lim_{x\to 0^+}\frac{\frac{d}{dx}\{\ln[\sin(x)]\}}{\frac{d}{dx}(x^{-1})}}$ ← using L'Hôpital's Rule

$\ln(y) = \lim_{x\to 0^+}\left[\frac{\frac{1}{\sin(x)}\cdot\cos(x)}{-1\cdot x^{-2}}\right]$

$\ln(y) = \lim_{x\to 0^+}\left[\frac{\cot(x)}{\frac{-1}{x^2}}\right]$ → Can we rewrite to find this? → $\boxed{\frac{0}{0}}$ Stop! Indeterminate Form

$\ln(y) = -\lim_{x\to 0^+}\left[\frac{\cot(x)}{\frac{1}{x^2}}\right]\cdot\left[\frac{x^2\tan(x)}{x^2\tan(x)}\right]$

$\ln(y) = -\lim_{x\to 0^+}\left[\frac{x^2}{\tan(x)}\right] \rightarrow \frac{(0)^2}{\tan(0)} = \boxed{\frac{0}{0}}$ Stop! Indeterminate Form

Ex.9 continued

$$\ln(y) = -\lim_{x \to 0^+} \frac{\frac{d}{dx}[x^2]}{\frac{d}{dx}[\tan(x)]}$$

Using L'Hôpital's Rule

$$\ln(y) = -\lim_{x \to 0^+} \frac{2x}{\sec^2(x)}$$

$$\ln(y) = -\left[\frac{2(0)}{\sec^2(0)}\right]$$

$$\ln(y) = 0$$

$$e^{\ln(y)} = e^0$$

$$y = 1$$

This ending is typical!!

Since $y = \lim_{x \to 0^+} [\sin(x)]^x$

$$1 = \lim_{x \to 0^+} [\sin(x)]^x$$

and

$$\lim_{x \to 0^+} [\cos(\tfrac{\pi}{2} - x)]^x = 1$$